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LETTER TO THE EDITOR

Integrable ergodic W^* systems

Anton Amann
Laboratory of Physical Chemistry, ETH-Zentrum, CH-8092 Zürich, Switzerland

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Abstract. A W^* system $(\mathcal{M}, G, \alpha)$ consists of a W^* algebra $\mathcal{M}$ (the 'algebra of observables'), a kinematical (locally compact, separable) group G and a representation α of G by automorphisms of $\mathcal{M}$. Ergodic W^* systems are used in algebraic quantum mechanics to describe elementary physical systems. An ergodic W^* system is integrable in the sense of Connes and Takesaki if it admits 'observables', i.e. operators which transform 'suitably' under the action α. Elementary quantum systems, corresponding to irreducible ray representations of G on a Hilbert space, and elementary classical systems, corresponding to transitive representations of G on a phase space, fit into this scheme. This letter investigates the structure of integrable ergodic W^* systems, in particular those of Abelian groups and those whose underlying W^* algebra is of type I.

Group-theoretical methods have played an important role in quantum mechanics from its early days (cf [1-6]). In particular, the structural understanding of the quantum mechanical formalism has profitted much from results such as von Neumann's uniqueness theorem ([7], theorem VIII.14), the investigation of representations of the rotation ([6], IV.6), the Lorentz [2] and the Galilei group [4].

To a large extent, group-theoretical methods can also be applied to classical mechanics [8, 9]. In a similar way one can characterise systems or particular physical quantities and gain insight into structural problems.

At first sight, the ingredients of quantum and classical theories are entirely different. For example, in quantum mechanics states are given by rays in a Hilbert space; in classical mechanics they correspond to points of a manifold (the phase space). Nevertheless, a closer inspection reveals that there are similarities and that a common structural setting can be found. In both cases physical quantities ('observables') are described by elements of an algebra, the algebra $\mathcal{B}(\mathcal{H})$ of bounded linear operators on a Hilbert space $\mathcal{H}$ or the algebra $\mathcal{L}_\infty(\Omega)$ of functions on the underlying manifold Ω. Group representations on the Hilbert space or on the manifold can be replaced by representations on the associated algebras through symmetries (automorphisms): If $u: G \ni g \to u(g) \in \mathcal{B}(\mathcal{H})$ is a unitary ray representation of the group G or $s: G \ni g \to s_g$, $s_g: \Omega \to \Omega$ is a representation of G (e.g. the Galilei group) on the manifold Ω, the associated representations $\{\alpha_g | q \in G\}$ by automorphisms of $\mathcal{B}(\mathcal{H})$ and $\mathcal{L}_\infty(\Omega)$, respectively, are given by

$$\alpha_g(x) \stackrel{\text{def}}{=} u(g) x u(g)^* \qquad x \in \mathcal{B}(\mathcal{H}), g \in G$$

$$(\alpha_g(f))(\omega) \stackrel{\text{def}}{=} f(s_{g^{-1}}(\omega)) \qquad f \in \mathscr{L}_\infty(\Omega), \omega \in \Omega, g \in \mathrm{G}.$$

Instead of studying group representations on a Hilbert space or on a manifold, one can now operate on a higher level: the investigation of group representations acting through automorphisms on algebras incorporates both classical and quantum theories among other things. Within the algebraic frame, systems with both quantum and classical properties or infinite systems can be described.

Of course, the development of group-theoretical methods in this extended form presents new difficulties, which have been only partially solved (cf [10-16]) with the help of mathematical techniques introduced in the last two decades.

In the following we restrict ourselves to a particular class of algebras, namely W^* algebras. This class is large enough to comprehend the algebras $\mathscr{B}(\mathscr{H})$ and $\mathscr{L}_\infty(\Omega)$ introduced above for quantum and classical mechanics.

A W^* algebra is a * algebra which is * isomorphic to a * algebra $\mathscr{R} \subseteq \mathscr{B}(\mathscr{H})$ of operators on a Hilbert space $\mathscr{H}$ fulfilling $\mathscr{R} = (\mathscr{R}')'$ (i.e. a von Neumann algebra). Here the commutant $\mathscr{S}'$ of a set $\mathscr{S}$ of operators on $\mathscr{H}$ is defined as

$$\mathscr{S}' \stackrel{\text{def}}{=} \{x \in \mathscr{B}(\mathscr{H}) | xy = yx, \forall y \in \mathscr{S}\}.$$

If W^* algebras $\mathscr{M}_i$, $i = 1, 2$, are * isomorphic, this is denoted by $\mathscr{M}_1 \simeq \mathscr{M}_2$.

Every W^* algebra $\mathscr{M}$ contains a unit element $\mathbf{1}$. A state ω on a W^* algebra $\mathscr{M}$ is a positive, linear, normalised $(\omega(1) = 1)$ mapping $\omega : \mathscr{M} \to \mathbb{C}$. A state ω is normal or σ-weakly continuous if $\sup_\beta \omega(x_\beta) = \omega(\sup_\beta x_\beta)$ holds for every bounded increasing net $(x_\beta)_{\beta \in I}$ of positive operators of $\mathscr{M}$ (I denotes an index set, sup the supremum).

The elements of $\mathscr{L}_\infty(\Omega) \stackrel{\text{def}}{=} \mathscr{L}_\infty(\Omega, \mu)$ are classes of essentially bounded Borel measurable complex-valued functions on Ω, essentially bounded with respect to a measure μ; for example, the Liouville measure. Two functions belong to the same class if they differ only on a μ-null set.

The centre $\mathscr{Z}(\mathscr{M})$ of a W^* algebra $\mathscr{M}$, defined as $\mathscr{Z}(\mathscr{M}) \stackrel{\text{def}}{=} \mathscr{M} \cap \mathscr{M}'$ contains the classical properties of the respective system. $\mathscr{M}$ is a factor if $\mathscr{Z}(\mathscr{M}) = \mathbb{C} \cdot \mathbf{1}$. Factors describe purely quantum mechanical systems (e.g. $\mathscr{M} \simeq \mathscr{B}(\mathscr{H})$). Commutative algebras (where $\mathscr{M} = \mathscr{Z}(\mathscr{M})$ holds, e.g. $\mathscr{M} \simeq \mathscr{L}_\infty(\Omega)$) describe purely classical systems.

A triple $(\mathscr{M}, \mathrm{G}, \alpha)$, consisting of a W^* algebra $\mathscr{M}$, a locally compact (kinematical) group G and a continuous representation $\alpha : \mathrm{G} \to \operatorname{Aut} \mathscr{M}$ of G into the automorphism group $\operatorname{Aut} \mathscr{M}$ of $\mathscr{M}$, is called a W^* system. Here continuity means that the mappings $\mathrm{G} \ni g \to \psi(\alpha_g(x)) \in \mathbb{C}$ are continuous for every $x \in \mathscr{M}$ and every normal state ψ on $\mathscr{M}$. Two W^* systems $(\mathscr{M}_i, \mathrm{G}, \alpha_i)$, $i = 1, 2$, are called conjugate, $(\mathscr{M}_1, \mathrm{G}, \alpha_1) \simeq (\mathscr{M}_2, \mathrm{G}, \alpha_2)$, if there exists a * isomorphism $J : \mathscr{M}_1 \to \mathscr{M}_2$, such that $\alpha_{2g} \circ J = J \circ \alpha_{1g}$, $g \in \mathrm{G}$, holds.

The locally compact group G will be assumed to be second countable and the W^* algebra $\mathscr{M}$ will be supposed to have a separable predual ($\mathscr{M}$ can then be faithfully represented on a separable Hilbert space).

A W^* system $(\mathscr{M}, \mathrm{G}, \alpha)$ is called ergodic if $\alpha_g(x) = x, \forall g \in \mathrm{G}$, implies $x = c \cdot \mathbf{1}$, $c \in \mathbb{C}$. The W^* system $(\mathscr{B}(\mathscr{H}), \mathrm{G}, \{\alpha_g(\ ) = u(g) \cdot u(g)^* | g \in \mathrm{G}\})$ is ergodic if and only if u is irreducible ([17], theorem 67.2). Furthermore, a W^* system implemented by a transitive phase-space representation is ergodic. Since 'irreducible' (cf [18]) and 'transitive' [8, 19] are the critieria for elementary quantum and classical systems, ergodic W^* systems can be regarded as elementary systems.

A W^* system $(\mathscr{M}, \mathrm{G}, \alpha)$ is called integrable if the set $\{\mathrm{x} \in \mathscr{M} | \int_{\mathrm{G}} \alpha_g(x^*x)\, \mathrm{d}g < \infty\}$ is σ-weakly dense in $\mathscr{M}$ [20]. $\int_{\mathrm{G}} \alpha_g(x^*x)\, \mathrm{d}g < \infty$ means that there is a $y \in \mathscr{M}$ such that $\psi(y) = \int_{\mathrm{G}} \psi(\alpha_g(x^*x))\, \mathrm{d}g$ holds for all normal states ψ on $\mathscr{M}$. Here $\mathrm{d}g$ denotes the left

Haar measure on G. W^* systems of a compact group are automatically integrable. The W^* system $(\mathcal{B}(\mathcal{H}), G, \{\alpha_g(\ ) = u(g)\cdot u(g)^* | g\in G\})$ is integrable if and only if the irreducible ray representation u is square integrable (cf [21]).

Every group G acts naturally ('from the left' or 'from the right') on the W^* algebra $\mathcal{L}_\infty(G) \overset{\text{def}}{=} \mathcal{L}_\infty(G, dg)$:

$$(\mathrm{Ad}\,\lambda(g)f)(s) \overset{\text{def}}{=} f(g^{-1}s) \qquad s, g\in G$$

$$(\mathrm{Ad}\,\rho(g)f)(s) \overset{\text{def}}{=} f(sg) \qquad s, g\in G$$

where $(\mathcal{L}_\infty(G), G, \mathrm{Ad}\,\lambda)$ and $(\mathcal{L}_\infty(G), G, \mathrm{Ad}\,\rho)$ are W^* systems.

An ergodic W^* system $(\mathcal{M}, G, \alpha)$ is integrable if and only if there exists a positive, linear, normal mapping $\chi : \mathcal{L}_\infty(G) \to \mathcal{M}$ with $\alpha_g \circ \chi = \chi \circ \mathrm{Ad}\,\lambda(g), g\in G$ [22] (χ is normal if $\psi\circ\chi$ is a normal state on $\mathcal{L}_\infty(G)$ for every normal state ψ on $\mathcal{M}$). Such a mapping (which is essentially a covariant semispectral measure, cf [23]) describes 'observables' in a system, defined as operators in $\mathcal{M}$, which transform 'suitably' under the representation α, i.e. just as a function f on G under $\mathrm{Ad}\,\lambda$ [22].

This concept of observables can be defined with respect to arbitrary (locally compact) groups G. If G is the Galilei group, observables are, for example, position, momentum, spin, etc. If $G = \mathbb{R}^2$ (used in von Neumann's uniqueness theorem), which can be regarded as a caricature of the Galilei group, integrability of an ergodic W^* system $(\mathcal{M}, \mathbb{R}^2, \alpha)$ is equivalent to the existence of (unbounded) operators Q and P, 'position' and 'momentum' (affiliated with $\mathcal{M}$), such that

$$\alpha_{(a,b)}(Q) = Q - a\cdot 1$$

$$\alpha_{(a,b)}(P) = P - b\cdot 1 \qquad a, b\in\mathbb{R}.$$

To summarise, ergodic integrable W^* systems describe elementary physical systems with observables.

Now consider a W^* system $(\mathcal{F}, H, \gamma)$ of a closed subgroup H of G. The induced W^* system $(\mathcal{M}, G, \alpha) = \mathrm{Ind}_H^G\{\mathcal{F}, \gamma\}$ is defined by

$$\mathcal{M} \overset{\text{def}}{=} \{y\in\mathcal{L}_\infty(G)\bar{\otimes}\mathcal{F} | (\mathrm{Ad}\,\rho(h)\otimes\gamma_h)(y) = y, \forall h\in H\}$$

$$\alpha_g(y) \overset{\text{def}}{=} (\mathrm{Ad}\,\lambda(g)\otimes\mathrm{Id})(y) \qquad y\in\mathcal{M}, g\in G.$$

Here $\bar{\otimes}$ denotes the W^* tensor product. Id is the identity mapping on $\mathcal{F}$ and $\mathrm{Ad}\,\lambda(g)\otimes\mathrm{Id}$ denotes the tensor product mapping of $\mathrm{Ad}\,\lambda(g)$ and Id. We now have the following theorem [24].

Theorem 1. Let $(\mathcal{M}, G, \alpha)$ be an integrable ergodic W^* system. Then there exist a closed subgroup H of G, a factor $\mathcal{F}$ and an integrable ergodic representation γ of H on $\mathcal{F}$ with $(\mathcal{M}, G, \alpha) \simeq \mathrm{Ind}_H^G\{\mathcal{F}, \gamma\}$. In particular, one has: $\mathcal{M} \simeq \mathcal{L}_\infty(G/H)\bar{\otimes}\mathcal{F}$, $(\mathcal{Z}(\mathcal{M}), G, \alpha|\mathcal{Z}(\mathcal{M})) \simeq (\mathcal{L}_\infty(G/H), G, \mathrm{Ad}\,\lambda_{G/H})$. Here G/H is the space of left cosets of H in G. $\mathcal{L}_\infty(G/H)$ is defined with respect to the unique G-invariant measure class on G/H and $(\mathrm{Ad}\,\lambda_{G/H}(g)f)(g_0H) \overset{\text{def}}{=} f(g^{-1}g_0H)$, $g, g_0\in G, f\in\mathcal{L}_\infty(G/H)$.

Due to theorem 1 it is sufficient to investigate ergodic integrable W^* systems whose underlying W^* algebra is a factor. The W^* algebra $\mathcal{M}$ of an integrable ergodic W^* system can be regarded as composed of a purely classical system (with W^* algebra

$\mathscr{L}_\infty(G/H)$) and a purely quantum mechanical system (with W^* algebra $\mathscr{F}$). For compact groups G theorem 1 was proved in [10] (cf [15]).

Now we have the following theorem ([22], theorem III.3).

Theorem 2. Let $(\mathscr{M}, G, \alpha)$ be an integrable ergodic W^* system. Then there exist a W^* system $(\mathscr{N}, G, \beta)$, a normal isomorphism $\pi: \mathscr{L}_\infty(G) \to \mathscr{N}$ of $\mathscr{L}_\infty(G)$ into $\mathscr{N}$ with $\beta_g \circ \pi = \pi \circ \mathrm{Ad}\, \lambda(g)$, $g \in G$, and an atomic projection p in the fixed point algebra $\mathscr{N}^\beta = \{y \in \mathscr{N} | \beta_g(y) = y, \forall g \in G\}$, such that $(\mathscr{M}, G, \alpha) \simeq (p\mathscr{N}p, G, \beta|_{p\mathscr{N}p})$ and $\pi(\mathscr{L}_\infty(G))$ is maximal commutative in $\mathscr{N}$, i.e. $\pi(\mathscr{L}_\infty(G))' \cap \mathscr{N} = \pi(\mathscr{L}_\infty(G))$.

The original W^* system $(\mathscr{M}, G, \alpha)$ is determined by the auxiliary system $(\mathscr{N}, G, \beta)$. The latter can be investigated by using the duality theory of W^* systems [25]. The covariant representation π, for example, gives rise to an ergodic coaction of G on $\mathscr{N}^\beta$ such that $(\mathscr{N}, G, \beta) \simeq (\mathscr{N}^\beta x_\delta G, G, \hat{\delta})$, where $\mathscr{N}^\beta x_\delta G$ is the crossed product of $\mathscr{N}^\beta$ by G with respect to δ and $\hat{\delta}$ is the dual action of δ (cf [25]). For Abelian groups G the above result can be used to develop a complete structure theory ([2], ch III.4).

Next consider the following theorem ([16] and [22], theorem III.6).

Theorem 3. Let $(\mathscr{M}, G, \alpha)$ be a faithful ergodic W^* system of the (LCS) Abelian group G (i.e. $\alpha_g = \mathrm{Id}$ implies $g = e$, where e is the unit element of G). Then $(\mathscr{M}, G, \alpha)$ is integrable if and only if for every element γ of the dual group $\hat{G}$ there exists a unitary u_γ with

$$\alpha_t(u_\gamma) = \langle \gamma, t \rangle u_\gamma \qquad t \in G, \gamma \in \hat{G}.$$

Here $\langle \cdot, \cdot \rangle : \hat{G} \times G \to \mathscr{T} = \{z \in \mathbb{C} | |z| = 1\}$ denotes the dual pairing between $\hat{G}$ and G. The unitaries u_γ, $\gamma \in \hat{G}$, are called eigenoperators of α.

$\hat{G} \ni \gamma \to u_\gamma \in \mathscr{M}$ can be regarded as a (Borel) ray representation with 2-cocycle (multiplier) $\sigma: \hat{G} \times \hat{G} \to \mathscr{T}$:

$$u_{\gamma_1} \cdot u_{\gamma_2} = \sigma(\gamma_1, \gamma_2) u_{\gamma_1 \gamma_2} \qquad \gamma_1, \gamma_1, \gamma_2 \in \hat{G}.$$

2-cocycles σ_i, $i = 1, 2$, are called equivalent, $\sigma_1 \sim \sigma_2$, if there exists a (Borel) function $d: \hat{G} \to \mathscr{T}$ with

$$\sigma_1(\gamma_1, \gamma_2) = \mathrm{d}(\gamma_1)\, \mathrm{d}(\gamma_2)\, \mathrm{d}(\gamma_1 \gamma_2)^* \sigma_2(\gamma_1, \gamma_2) \qquad \gamma_1, \gamma_2 \in \hat{G}.$$

The set of 2-cocycles modulo the equivalence relation $\sim$ is a group, the second cohomology group $H^2(\hat{G}, \mathscr{T})$.

Now consider the following theorem ([16] and [22], theorem III.5).

Theorem 4. Let G be a (LCS) Abelian group. Then for every $\tilde{\sigma} \in H^2(\hat{G}, \mathscr{T})$ there exists (up to isomorphism) exactly one integrable ergodic W^* system $(\mathscr{M}, G, \alpha)$ with eigenoperators $\{u_\gamma | \gamma \in \hat{G}\}$, such that an associated 2-cocycle lies in $\tilde{\sigma}$. In particular, the W^* algebra $\mathscr{M}$ is generated by the unitaries $\{u_\gamma | \gamma \in \hat{G}\}$.

Remark. The results of the following theorem ([22], theorem III.7) are well known for compact Abelian groups [10, 11].

Theorem 5. Let $(\mathcal{M}, \mathrm{G}, \alpha)$ be a faithful integrable ergodic W^* system of the (LCS) Abelian group G and $\sigma: \hat{\mathrm{G}} \times \hat{\mathrm{G}} \to \mathcal{T}$ be an associated (Borel) 2-cocycle. Then

$$\sigma(\gamma_0, \gamma)^* \cdot \sigma(\gamma, \gamma_0) = \langle \gamma_0, \iota(\gamma) \rangle \qquad \gamma, \gamma_0 \in \hat{\mathrm{G}}$$

defines a continuous group homomorphism $\iota: \hat{\mathrm{G}} \to \mathrm{G}$, such that

(i) $\mathcal{M}$ is of type I $\Leftrightarrow \iota(\hat{\mathrm{G}}) = \overline{\iota(\hat{\mathrm{G}})}$, i.e. $\iota(\hat{\mathrm{G}})$ is closed

(ii) $\mathcal{M}$ is a factor $\Leftrightarrow \overline{\iota(\hat{\mathrm{G}})} = \mathrm{G}$

(iii) $\mathcal{M}$ is finite $\Leftrightarrow \overline{\iota(\hat{\mathrm{G}})}$ is compact.

Furthermore, the factor $\mathcal{F}$ in the tensor product decomposition $\mathcal{M} \simeq \mathcal{L}_\infty(\mathrm{G}/\mathrm{H}) \bar{\otimes} \mathcal{F}$ (cf theorem 1) is always semifinite and injective. In particular, if $\mathcal{F}$ is not of type I (i.e. not isomorphic to an algebra $\mathcal{B}(\mathcal{H})$ for a suitable Hilbert space $\mathcal{H}$), it is either isomorphic to the unique injective type II_1 factor or isomorphic to the unique injective type II_∞ factor (cf [26]).

Example. Consider $\mathrm{G} = \mathbb{R}^2$. Every 2-cocycle of $\hat{\mathbb{R}}^2 \simeq \mathbb{R}^2$ is of the form

$$\sigma_c((t_0, s_0), (t, s)) = \exp\{ic(t_0 s - s_0 t)\} \qquad t_0, s_0, t, s \in \mathbb{R}$$

for a suitable number $c \in \mathbb{R}$ ([27], theorem 5.4). For the associated group homomorphism $\iota: \hat{\mathbb{R}}^2 \to \mathbb{R}^2$ one gets

$$\iota(t, s) = 2c(-t, s) \qquad t, s \in \mathbb{R}.$$

Thus either $\overline{\iota(\hat{\mathbb{R}}^2)} = \iota(\hat{\mathbb{R}}^2) = \mathbb{R}^2$ (if $c \neq 0$) or $\iota(\hat{\mathbb{R}}^2) = (0, 0)$. For $c \neq 0$ one gets the well known type I factor systems (which correspond to an irreducible ray representation of the Weyl relations), for $c = 0$ the commutative W^* system $(\mathcal{L}_\infty(\mathbb{R}^2), \mathbb{R}^2, \mathrm{Ad}\,\lambda)$. The former correspond to the simplest version of quantum mechanics, the latter to the simplest version of a classical theory. Note that these results generalise von Neumann's uniqueness theorem ([7], theorem VIII.14).

Now consider the following theorem ([22], lemma III.5).

Theorem 6. Let $(\mathcal{M}, \mathrm{G}, \alpha)$ be an integrable ergodic W^* system, where $\mathcal{M}$ is a type I factor, and consider an auxiliary system $(\mathcal{N}, \mathrm{G}, \beta)$ (theorem 2) with the covariant representation $\pi: \mathcal{L}_\infty(\mathrm{G}) \to \mathcal{N}$. Then there exist unitaries $n(g)$, $g \in \mathrm{G}$, in $\mathcal{N}^\beta$ with the property

$$n(g)\pi(f)n(g)^* = \pi(\mathrm{Ad}\,\rho(g)f) \qquad f \in \mathcal{L}_\infty(\mathrm{G}), g \in \mathrm{G}.$$

The unitaries $n(g)$, $g \in \mathrm{G}$, are determined uniquely up to a complex number of modulus 1 (this is due to the maximal commutativity of $\pi(\mathcal{L}_\infty(\mathrm{G}))$ in $\mathcal{N}$). They form a ray representation $\mathrm{G} \ni g \to n(g) \in \mathcal{N}^\beta$ with a 2-cocycle $c: \mathrm{G} \times \mathrm{G} \to \mathcal{T}$. The equivalence class $\tilde{c} \in \mathrm{H}^2(\mathrm{G}, \mathcal{T})$ of c specifies completely the auxiliary system $(\mathcal{N}, \mathrm{G}, \beta)$ (see [22], ch III.5, and [21]). For more information about integrable ergodic actions on type I factors, see [21]. The converse of theorem 6 is as follows.

Theorem 7. Let $(\mathcal{M}, \mathrm{G}, \alpha)$ be an integrable ergodic W^* system with $\mathcal{M}$ a factor and consider au auxiliary system $(\mathcal{N}, \mathrm{G}, \beta)$ (theorem 2) with the covariant representation $\pi: \mathcal{L}_\infty(\mathrm{G}) \to \mathcal{N}$. Suppose there exist unitaries $n(g)$, $g \in \mathrm{G}$, in $\mathcal{N}^\beta$ with the property $n(g)\pi(f)n(g)^* = \pi(\mathrm{Ad}\,\rho(g)f)$, $f \in \mathcal{L}_\infty(\mathrm{G})$, $g \in \mathrm{G}$.

Then $\mathcal{N}$ and consequently $\mathcal{M}$ are W^* algebras of type I, i.e. isomorphic to $\mathcal{B}(\mathcal{H})$ for a suitable Hilbert space $\mathcal{H}$.

Proof. The theorem follows essentially from ([28], theorem 6). To make use of the latter, one has to show that the mapping $G \ni g \to n(g) \in \mathcal{N}^\beta$ can be considered as Borel measurable. This follows from our separability assumptions (G second coutable, the predual of $\mathcal{M}$ separable) by standard techniques.

Consider an integrable ergodic W^* system $(\mathcal{M}, G, \alpha)$, where $\mathcal{M}$ is a factor, and an associated auxiliary system $(\mathcal{N}, G, \beta)$ (theorem 2) with covariant representation $\pi : \mathcal{L}_\infty(G) \to \mathcal{N}$. Let $U(\mathcal{N}^\beta)$ denote the unitaries of $\mathcal{N}^\beta$ and define

$$S \overset{\text{def}}{=} \{g \in G | \exists n(g) \in U(\mathcal{N}^\beta) \text{ with } n(g)\pi(f)n(g)^* = \pi(\text{Ad}\, \rho(g)f), f \in \mathcal{L}_\infty(G)\}.$$

Then S is a subgroup of G.

Conjecture 1. S is a dense normal subgroup of G.

Conjecture 1 is true for W^* systems $(\mathcal{M}, G, \alpha)$ where G is Abelian: in fact S is then given as $\iota(\hat{G})$ (cf [22], ch III.4 and see theorem 5(ii)). It is furthermore true for W^* systems $(\mathcal{M}, G, \alpha)$ where the factor $\mathcal{M}$ is of type I. In this case, even S = G holds (theorem 6). S = G is even characteristic for type I systems (theorem 7).

Conjecture 2. Let $(\mathcal{M}, G, \alpha)$ be an integrable ergodic W^* system, where $\mathcal{M}$ is a factor and G is a simple group, i.e. it does not admit normal subgroups besides G itself and the trivial subgroup $\{e\}$ formed by the unit element e. Then $\mathcal{M}$ is of type I.

Conjecture 2 is an immediate consequence of conjecture 1. It would generalise the result of Wassermann [29], which asserts that the rotation group SO(3) cannot act ergodically on type II_1 factors.

References

[1] Weyl H 1927 *Z. Phys.* **46** 1-46
[2] Wigner E 1939 *Ann. Math.* **40** 149-204
[3] Jauch J M 1968 *Foundations of Quantum Mechanics* (Reading, MA: Addison-Wesley)
[4] Lévy-Leblond J-M 1971 *Group Theory and its Applications* vol 2, ed E M Loebl (New York: Academic)
[5] Lévy-Leblond J-M 1974 *Riv. Nuovo Cimento* 4 99-143
[6] Jost R 1969 *Quantenmechanik I* (Zurich: ETH)
[7] Reed M and Simon B 1975 *Methods of Modern Mathematical Physics* vol 1 (New York: Academic)
[8] Souriau J-M 1970 *Structures des Systèmes Dynamiques* (Paris: Dunod)
[9] Jost R 1971 *Klassische Mechanik* (Zurich: ETH)
[10] Albeverio S and Hoegh-Krohn R 1980 *Math. Z.* **174** 1-17
[11] Olesen D, Pedersen G K and Takesaki M 1980 *J. Operator Theory* **3** 237-69
[12] Hoegh-Krohn, R Landstad M B and Størmer E 1981 *Ann. Math.* **114** 75-86
[13] Wassermann A J 1981 *Thesis* University of Pennsylvania
[14] DeSchreye D 1983 *Math. Scand.* **53** 265-80
[15] Landstad M B 1984 *Proc. Int. Conf. Neptun,* 1980 2 vols (Bucharest: Increst)
[16] DeSchreye D 1985 *Math. Scand.* **57** 189-205
[17] Berberian S K 1974 *Lectures in Functional Analysis and Operator Theory* (Berlin: Springer)
[18] Schwartz L 1968 *Application of Distributions to the Theory of Elementary Particles in Quantum Mechanics* (New York: Gordon and Breach)
[19] Arens R 1971 *Commun. Math. Phys.* **21** 139-49
[20] Connes A and Takesaki M 1977 *Tôhôku Math. J.* **29** 473-575
[21] Amann A 1986 *J. Math. Phys.* **27** 2282-9
[22] Amann A 1986 *Fortschr. Phys.* **34** 167-215

[23] Scutaru H 1977 *Lett. Math. Phys.* **2** 101-7
[24] Amann A 1987 in preparation
[25] Nakagami Y and Takesaki M 1979 *Duality for Crossed Products of von Neumann Algebras* (*Lecture Notes in Mathematics* **731**) (Berlin: Springer)
[26] Connes A 1976 *Ann. Math.* **104** 73-115
[27] Parthasarathy K R 1969 *Multipliers on Locally Compact Groups* (*Lecture Notes in Mathematics* **93**) (Berlin: Springer)
[28] Nakagami Y and Sutherland C E 1979 *Pacific J. Math.* **83** 221-9
[29] Wassermann A 1987 *Invent. Math.* to be published